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Probabilistic Coding of Quantum States of Different Dimensions

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Abstract We investigate the scheme which enables encoding of n qudits whose Hilbert spaces are of different dimensions to one qudit of dimension smaller than that of the Hilbert space of the n qudits and then probabilistically but error-free decoding any subset of k qudits. We also give a formula for calculating the average probability of successful decoding.

Keywords Probabilistic coding · Qudits

1 Introduction

Recently, Grudka and Wojcik proposed a scheme which enables encoding the states of two non-entangled qubits into one qutrit in a probabilistic, but error-free manner. Namely, the protocol enables decoding with average probability $2/3$ and perfect fidelity one arbitrarily chosen qubit after the encoding took place [1]. The deep reason for this lies in the same number of parameters specifies two non-entangled qubits in a pure state and one qutrit. Bertuskova et al. have shown that the protocol can be generalized to higher dimensions [2]. Specifically, they have shown how to encode n qudits of dimension d each in one qudit of dimension $n(d-1)+1$ and then decode one of them. Grudka et al. presented another protocol which enables encoding of N qudits into one qudit of dimension smaller than the Hilbert space of the original system and then probabilistically but error-free decode any subset of k $(n > k)$ qudits [3]. In these proposed protocols, the particle be encoded is same, that is to say, the dimension of the Hilbert space of the particle be encoded is same. But in our scheme, we investigate probabilistic coding of quantum states with two different quantum systems and generalized it to many different quantum systems.

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2 Coding of a Qubit and a Qutrit

Let us introduce two parties Alice and Bob. Alice performs the encoding while Bob tries to decode the qubit or qutrit with perfect fidelity. We suppose the states of the qubit and the qutrit are

$$|\psi_1\rangle = \alpha_1|0\rangle_1 + \beta_1|1\rangle_1 + \gamma_1|2\rangle_1 \tag{1}$$

and

$$|\psi_2\rangle = \alpha_2|0\rangle_2 + \beta_2|1\rangle_2, \tag{2}$$

respectively. To encode the states of the qubit and the qutrit to a four-dimensional qudit, Alice performs generalized measurement on the joint state of the system $|\psi_1\rangle \otimes |\psi_2\rangle$ given by the following six measurement operators

$$M_{ij} = \frac{1}{2}\left(|ij\rangle\langle ij| + \sum_{k=0,k\neq i}^{2} |kj\rangle\langle kj| + \sum_{l=0,l\neq j}^{1} |il\rangle\langle il|\right), \tag{3}$$

where $i = 0, 1, 2$; $j = 0, 1$. Explicitly,

$$M_{00} = \frac{1}{2}(|00\rangle\langle 00| + |10\rangle\langle 10| + |20\rangle\langle 20| + |01\rangle\langle 01|), \tag{4}$$

$$M_{01} = \frac{1}{2}(|01\rangle\langle 01| + |11\rangle\langle 11| + |21\rangle\langle 21| + |00\rangle\langle 00|), \tag{5}$$

$$M_{10} = \frac{1}{2}(|10\rangle\langle 10| + |00\rangle\langle 00| + |20\rangle\langle 20| + |11\rangle\langle 11|), \tag{6}$$

$$M_{11} = \frac{1}{2}(|11\rangle\langle 11| + |01\rangle\langle 01| + |21\rangle\langle 21| + |10\rangle\langle 10|), \tag{7}$$

$$M_{20} = \frac{1}{2}(|20\rangle\langle 20| + |00\rangle\langle 00| + |10\rangle\langle 10| + |21\rangle\langle 21|), \tag{8}$$

$$M_{21} = \frac{1}{2}(|21\rangle\langle 21| + |01\rangle\langle 01| + |11\rangle\langle 11| + |20\rangle\langle 20|). \tag{9}$$

If, as a result of the measurement, Alice obtains $(0, 0)$, then the joint state is projected onto a four dimensional subspace (a qudit) and is now given as

$$|\Psi\rangle = \frac{1}{\sqrt{N}}(\alpha_1\alpha_2|00\rangle + \beta_1\alpha_2|10\rangle + \gamma_1\alpha_2|20\rangle + \alpha_1\beta_2|01\rangle), \tag{10}$$

where $N = |\alpha_1\alpha_2|^2 + |\beta_1\alpha_2|^2 + |\gamma_1\alpha_2|^2 + |\alpha_1\beta_2|^2$ is the probability of outcome $(0, 0)$.

Now Bob can choose which qudit to be decoded. If Bob choose to decode the qutrit, he should performs a projective measurement given by the following operators:

$$P_{1,S} = |00\rangle\langle 00| + |10\rangle\langle 10| + |20\rangle\langle 20|, \tag{11}$$

$$P_{1,F} = |01\rangle\langle 01|. \tag{12}$$

If Bob obtains $(1, S)$, then the state of the qudit is projected onto three dimensional subspace and is identical to the state of the first qutrit given by (1). That is

$$|\Psi_{out}\rangle = \frac{\alpha_2}{\sqrt{N}}(\alpha_1|0\rangle + \beta_1|1\rangle + \gamma_1|2\rangle), \tag{13}$$

and the decoding is perfect. The probability of the successful decoding of the qutrit is $|\alpha_2|^2/N$. If Bob obtains $(1, F)$, then the procedure of decoding fails.

Similarly, to decode the qubit, Bob performs projective measurement described by the operators:

$$P_{2,S} = |00\rangle\langle 00| + |01\rangle\langle 01|, \tag{14}$$

$$P_{2,F} = |10\rangle\langle 10| + |20\rangle\langle 20|, \tag{15}$$

and the probability of the successful decoding of the qubit is $|\alpha_1|^2/N$.

3 Coding of Two Different Qudits

We suppose that we have two non-entangled qudits of the dimension s and d, respectively. Each of them is in a pure state and the joint state is

$$|\Phi\rangle = \sum_{i=0}^{s-1} \alpha_i |i\rangle \otimes \sum_{i=0}^{d-1} \beta_i |i\rangle. \tag{16}$$

To encode the states of these different qudits into one qudit of dimension $s + d - 1$ Alice performs generalized measurement on the state of the system $|\Phi\rangle$ described by the following sd operators:

$$M_{i,j} = \frac{1}{\sqrt{s+d-1}}\left(|ij\rangle\langle ij| + \sum_{k=0,k\neq i}^{s-1} |kj\rangle\langle kj| + \sum_{l=0,l\neq j}^{d-1} |il\rangle\langle il|\right). \tag{17}$$

That normalized state of the system after the encoding with $M_{i,j}$ reads

$$|\Phi_{i,j}\rangle = \frac{1}{\sqrt{N_{i,j}}}\left(\alpha_i\beta_j|ij\rangle + \sum_{k=0,k\neq i}^{s-1} \alpha_k\beta_j|kj\rangle + \sum_{l=0,l\neq j}^{d-1} \alpha_i\beta_l|il\rangle\right), \tag{18}$$

where $N_{i,j} = |\alpha_i\beta_j|^2 + \sum_{k=0,k\neq i}^{s-1} |\alpha_k\beta_j|^2 + \sum_{l=0,l\neq j}^{d-1} |\alpha_i\beta_l|^2$ is the probability.

In the procedure of decoding the first qudit, the state of the system will be project onto an s-dimensional subspace and the projectors corresponding to the decoding of the first qudit read

$$P_{1,S} = \sum_{k=0}^{s-1} |kj\rangle\langle kj|, \tag{19}$$

$$P_{1,F} = \sum_{l=0,l\neq j}^{d-1} |il\rangle\langle il|. \tag{20}$$

Projection onto $P_{1,S}$ indicates successful decoding of the first qudit while $P_{1,F}$ signals failure. The probability of the successful decoding of the first qudit is $|\beta_j|^2/N$. The procedure for decoding of the second qudit is similar.

4 Coding of Many Different Qudits

In this section we investigate the scheme for encoding of n qudits in such a way that one can decode any subset of k qudits probabilistically, but error-free. Suppose that we have n non-entangled qudits of dimension $d_1, d_2, \ldots, d_n$, respectively. Each of these qudits is in a pure state and the state of the joint system is

$$|\Upsilon\rangle = \sum_{i=0}^{d_1-1} \alpha_{1i}|i\rangle \otimes \sum_{i=0}^{d_2-1} \alpha_{2i}|i\rangle \otimes \cdots \otimes \sum_{i=0}^{d_n-1} \alpha_{ni}|i\rangle. \tag{21}$$

To encode these n qudits Alice performs a generalized measurement described by the following measurement operators:

$$\begin{aligned} M_{i,j,l,\ldots} = \frac{1}{\sqrt{D_k}}\Bigg(& |ijl\cdots\rangle\langle ijl\cdots| + \sum_{p=0,p\neq i}^{d_1-1} |pjl\cdots\rangle\langle pjl\cdots| \\ & + \sum_{q=0,q\neq j}^{d_2-1} |iql\cdots\rangle\langle iql\cdots| + \sum_{r=0,r\neq l}^{d_3-1} |ijr\cdots\rangle\langle ijr\cdots| + \cdots \\ & + \sum_{p=0,p\neq i}^{d_1-1}\sum_{q=0,q\neq j}^{d_2-1} |pql\cdots\rangle\langle pql\cdots| + \sum_{p=0,p\neq i}^{d_1-1}\sum_{r=0,r\neq l}^{d_3-1} |pjr\cdots\rangle\langle pjr\cdots| \\ & + \sum_{q=0,q\neq j}^{d_2-1}\sum_{r=0,r\neq l}^{d_3-1} |iqr\cdots\rangle\langle iqr\cdots| + \text{other terms}\Bigg), \end{aligned} \tag{22}$$

where “other terms” denotes similar sums over three, four $\cdots$ and k indices. The constant D_k is equal to the dimension of the subspace onto which $M_{i,j,l,\ldots}$ projects and has the form

$$\begin{aligned} D_k = {} & 1 + \sum_i (d_i - 1) + \sum_{i\neq j} (d_i - 1)(d_j - 1) \\ & + \sum_{i\neq j\neq l} (d_i - 1)(d_j - 1)(d_l - 1) + \text{other terms}, \end{aligned} \tag{23}$$

where “other terms” denotes similar sums of all possible terms with four, five, $\cdots$ and k distinct factors like $(d_i - 1)$.

In order to decode k qudits, Bob should perform a projective measurement. The set of measurement operators depends on the choice of k qudits to be decoded. For example, if Alice obtains the result $(i_1, i_2, \ldots, i_n)$, and Bob want to decode the first k qudits, the measurement operators are

$$P_S = \sum_{p_1=0}^{d_1-1}\sum_{p_2=0}^{d_2-1}\cdots\sum_{p_k=0}^{d_k-1} |p_1 p_2 \cdots p_k i_{k+1} \cdots i_n\rangle\langle p_1 p_2 \cdots p_k i_{k+1} \cdots i_n|, \tag{24}$$

$$P_F = 1 - P_S. \tag{25}$$

If we suppose that each qudit is prepared in a randomly chosen pure state then the average probability of successful decoding of k qudits of n encoded qudits is

$$p_s = \frac{1}{D_k} \prod_{j=1}^{k} d_i^j. \tag{26}$$

This is the dimension of the Hilbert space of k decoded qudits divided by the dimension of the Hilbert space of the qudit to which n qudits were encoded.

5 Summary

In this letter we have investigated probabilistic codings with two and many different qudits. Our protocol enables encoding of many qudits in one qudit of dimension smaller than that of the Hilbert space of the original system and then decode some subset of them probabilistically. Our protocol is more general, because the Hilbert spaces of these qudits are of different dimensions. We hope that the protocol can be completed by experiments.

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